Mensuration: Lengths-Areas-Volumes

This document describes a lesson plan which could be used to introduce the Mensuration section of the Grade 11 Mathematics syllabus in Zambia. It aims to build an understanding of concepts in the students, along with some useful rules to remember, rather than teaching all of the formulae needed for the exams. I would suggest that students do not take notes, rather that they listen, ask questions, and try to understand. Future lessons will be needed in which formulae are learnt, example questions are attempted and the students practice using their skills.

Duration: around 80 minutes

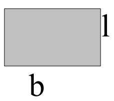
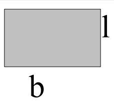
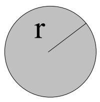
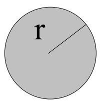
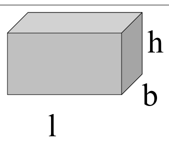
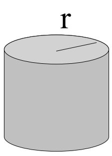
Equipment: Chalkboard and chalk.

Assumption: students have met some formulae for areas and lengths.

Units

Many students seem to get confused by whether they should use m^2, cm^3 or whatever. They also sometimes confuse formulae for volume/area/length. This discussion is designed to review and reinforce the difference between these concepts.
Construct this table on the board, telling the students what they are about to review. Just add one line at a time and add the grey information through discussion with the students.

Each time reinforce the idea that areas need two lengths multiplied together, volumes: 3 and so on.

Shape	Formula	Number of lengths multiplied together	Units of result	Length / Area / Volume
	2l+2b	1 (2 is not a length!)	cm	Length (perimeter)
	lb	2 (l times b)	cm^2	Area
	πr^2	2 (r times r)	cm^2	Area
	$2\pi r$	1	cm	Length (circumference)
	lbh	3	cm^3	Volume
	$\pi r^2 h$	3	cm^3	Volume

At this stage, check that the students have identified the pattern that the number in column 3 is the same as the exponent in column 4.

Now give the following equations – can the identify whether it gives an area or volume?

- $4\pi r^3/3$
- $4\pi r^2$

Can they guess what shape these formulae apply to?

Summarise again – if three lengths multiplied together, it must be a volume. If two, then it's an area. If one, it's a length.

Exploring 3 Dimensions

This section reintroduces the fundamentals of geometry – the three dimensions, and the origins of length, area and volume calculations. It probably uses a different approach from the standard syllabus, but this will help to cement the ideas more clearly in the students' minds.

What do we mean by dimension?

A dimension is a direction in which one can measure distance. For example, "horizontal" or "left and right" is a dimension. If we draw a horizontal line on the board, we can measure the distance from one end to the other – this is its length.

Length is the size in one dimension. It could be the amount of string needed to cover a line.

Going from right to left is travelling within the same dimension as travelling from left to right. They both use the same line, or one parallel to it.

We can call this the first dimension.

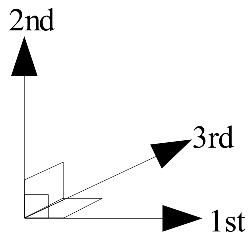
To find the next dimension, we have to rotate by 90 degrees. It's good to get a student up to draw a line at 90 degrees to the first one (I.e. it should be vertical!)

We only consider it to be a new dimension if it is perpendicular to the other dimensions. Now we have two dimensions on the board – we can go left-right, or up-down. The surface of the board is two-dimensional – we can move to any point on it by going some distance left or right, and some distance up and down. If we draw a two-dimensional shape on the board, we can measure the size of its surface – this is its area. Get the students to give some examples of two dimensional shapes.

Area is the size in two dimensions. It could be the amount of paper needed to cover a shape.

Now ask a student to come up and draw another line at right angles to both of the first two.

Of course, it's not possible, but point directly out of the board to indicate that if the first two dimensions are left-right and up-down, the third must be "forwards-backwards", or "in-out". We can't draw this properly on the board, but can represent it using a line between the first two as if we are looking at the corner of a cube from above and to one side of it.



One could show the students how to arrange thumb and first two fingers so that they represent the three dimensions.

Three dimensions allow movement throughout space – we can get from one point to another by going left or right, then up or down, then forwards or backwards.

If we draw a three-dimensional shape, we could measure the size of the space inside it – this is its volume.

Volume is the size in three dimensions. It could be the amount of liquid needed to fill a shape.

How do we find lengths/areas/volumes?

1-dimension

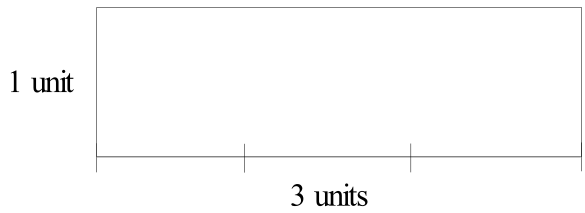
Draw a horizontal line. Ask the students how many dimensions it has. We can divide it up into sections, if we want, and can call each one "one unit".

This is a line in one dimension, whose length is 3 units.

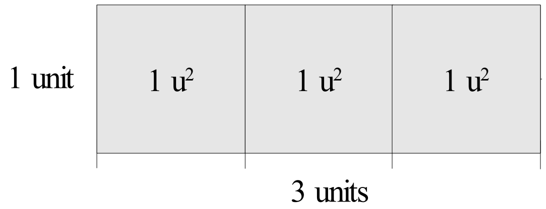
2-dimensions

Now we want the students to imagine that the line is actually a very thin roll of modelling clay (plasticine/play dough), and we're now going to stretch it out in the second dimension – I.e. upwards, as if we were making pastry.

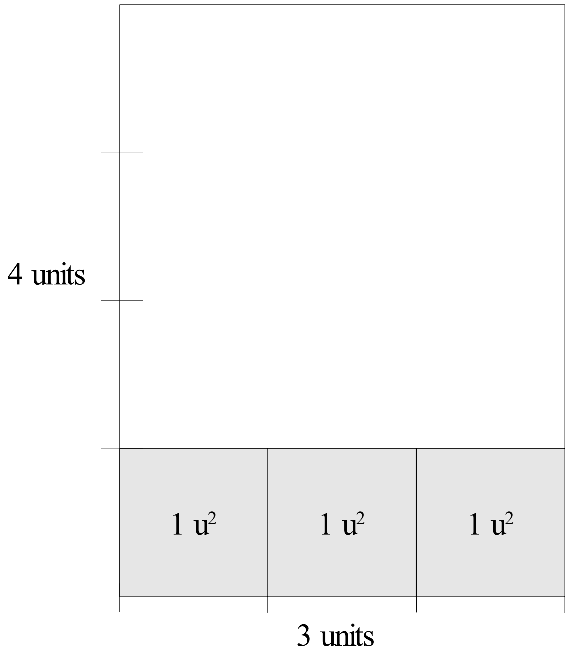
Let's stretch it by one unit. It will now look like this.



Students should be able to identify that it's a rectangle. The length along the base is 3 units, and along the height is 1 unit. What size is it's area? Well, let's define that as follows. A square with side length 1 unit is defined as having an area of 1 unit squared, or 1 u^2. Let's place one of these squares on top of each unit length of the line at the base of the rectangle.



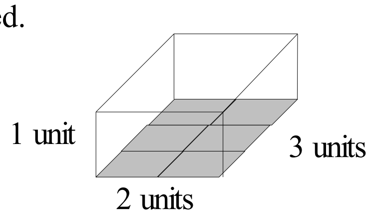
We can see that the total area of this rectangle must be 3 u^2.
Now suppose that we take this and stretch it upwards still further, until it is 4 units high.



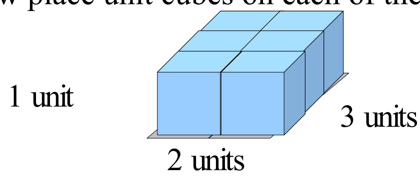
If we continue to place the unit squares into this shape, we can see that we will have a total of four rows like the bottom one, so there must be 4 times 3 = 12 units square of area in this rectangle. Because we can always build a rectangle by drawing a line some number of units long, placing that number of squares on top of it and extending upwards by some number of rows, the area can always be given by base times height.

3-dimensions

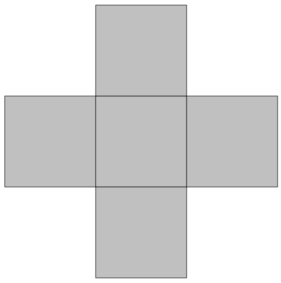
Explain that this modelling clay is very stretchy – we're going to take a 2 by 3 rectangle (as this makes the arithmetic easier) and stretch it out into the third dimension, just as we stretched the line into the second dimension. So, we'll draw our rectangle as being parallel to the floor and we're going to grab hold and pull it upwards by one unit, so that it keeps its rectangular shape and size, but becomes taller and box shaped.

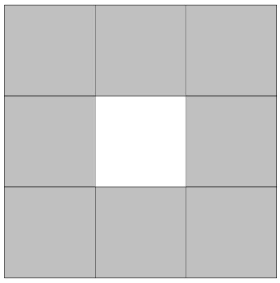


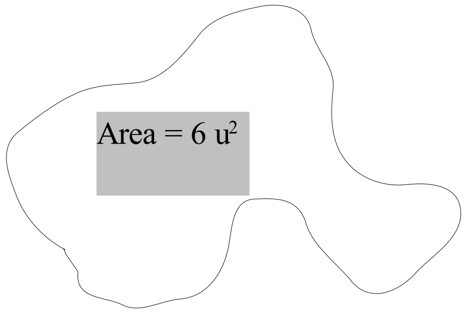
Now, what is the volume of this shape? We define the volume of a cube which is 1 unit on each side to be 1 unit cubed or 1 u^3. Suppose that, just as we placed unit squares on each section of the line to calculate an area, we now place unit cubes on each of the squares in the above shape.



Counting them shows that the volume must be 6 u^3. Now extend the drawing up by another unit. Placing unit cubes on the top of each of the ones in the layer below will give us a total volume of 2 times 6 = 12 u^3.
In fact, if we multiply the base area by the height, we get the volume. It's like adding floors to a building. This is true for all 3-dimensional shapes in which a constant area is maintained as the volume increases in the third dimension. Give the examples of some cross-sections like these:







They should now be able to understand why the volumes of cuboids, cylinders and prisms all follow similar forms of area times height.